ANTI-POWER PREFIXES OF THE THUE-MORSE WORD

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Abstract. Recently, Fici, Restivo, Silva, and Zamboni defined a k-anti-power to be a word of the form $w_1 w_2 \cdots w_k$, where $w_1, w_2, \ldots, w_k$ are distinct words of the same length. They defined $AP(x,k)$ to be the set of all positive integers m such that the prefix of length km of the word x is a k-anti-power. Let $\mathbf{t}$ denote the Thue-Morse word, and let $\mathcal{F}(k) = AP(\mathbf{t},k) \cap (2\mathbb{Z}^+ - 1)$. For $k \geq 3$, $\gamma(k) = \min(\mathcal{F}(k))$ and $\Gamma(k) = \max((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k))$ are well-defined odd positive integers. Fici et al. speculated that $\gamma(k)$ grows linearly in k. We prove that this is indeed the case by showing that $1/2 \leq \liminf_{k\to\infty}(\gamma(k)/k) \leq 9/10$ and $1 \leq \limsup_{k\to\infty}(\gamma(k)/k) \leq 3/2$. In addition, we prove that $\liminf_{k\to\infty}(\Gamma(k)/k) = 3/2$ and $\limsup_{k\to\infty}(\Gamma(k)/k) = 3$.

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1. Introduction

A well-studied notion in combinatorics on words is that of a k-power; this is simply a word of the form w^k for some word w. It is often interesting to ask questions related to whether or not certain types of words contain factors (also known as substrings) that are k-powers for some fixed k. For example, in 1912, Axel Thue [7] introduced an infinite binary word that does not contain any 3-powers as factors (we say such a word is cube-free). This infinite word is now known as the Thue-Morse word; it is arguably the world's most famous (mathematical) word [1, 2, 3, 4, 5].

Definition 1.1. Let $\overline{w}$ denote the Boolean complement of a binary word w. Let $A_0 = 0$. For each nonnegative integer n, let $B_n = \overline{A_n}$ and $A_{n+1} = A_n B_n$. The *Thue-Morse word* $\mathbf{t}$ is defined by

$$\mathbf{t} = \lim_{n\to\infty} A_n.$$

Recently, Fici, Restivo, Silva, and Zamboni [6] introduced the very natural concept of a k-anti-power; this is a word of the form $w_1 w_2 \cdots w_k$, where $w_1, w_2, \ldots, w_k$ are distinct words *of the same length*. For example, 001011 is a 3-anti-power, while 001010 is not. In [6], the authors prove that for all positive integers k and r, there is a positive integer $N(k,r)$ such that all words of length at least $N(k,r)$ contain a factor that is either a k-power or an r-anti-power. They also define

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$AP(x,k)$ to be the set of all positive integers m such that the prefix of length km of the word x is a k-anti-power. We will consider this set when $x = \mathbf{t}$ is the Thue-Morse word. It turns out that $AP(\mathbf{t},k)$ is nonempty for all positive integers k [6, Corollary 6]. It is not difficult to show that if k and m are positive integers, then $m \in AP(\mathbf{t},k)$ if and only if $2m \in AP(\mathbf{t},k)$. Therefore, the only interesting elements of $AP(\mathbf{t},k)$ are those that are odd. For this reason, we make the following definition.

Definition 1.2. Let $\mathcal{F}(k)$ denote the set of odd positive integers m such that the prefix of $\mathbf{t}$ of length km is a k-anti-power. Let $\gamma(k) = \min(\mathcal{F}(k))$ and $\Gamma(k) = \sup((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k))$.

Remark 1.1. It is immediate from Definition 1.2 that $\mathcal{F}(1) \supseteq \mathcal{F}(2) \supseteq \mathcal{F}(3) \supseteq \cdots$. Therefore, $\gamma(1) \leq \gamma(2) \leq \gamma(3) \leq \cdots$ and $\Gamma(1) \leq \Gamma(2) \leq \Gamma(3) \leq \cdots$.

For convenience, we make the following definition.

Definition 1.3. If m is a positive integer, let $\mathfrak{K}(m)$ denote the smallest positive integer k such that the prefix of $\mathbf{t}$ of length km is not a k-anti-power.

If $k \geq 3$, then $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is nonempty because it contains the number 3 (the prefix of $\mathbf{t}$ of length 9 is 011010011, which is not a 3-anti-power). We will show (Theorem 3.1) that $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is finite so that $\Gamma(k)$ is a positive integer for each $k \geq 3$. For example, $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(6) = \{1,3,9\}$. This means that $AP(\mathbf{t},6)$ is the set of all postive integers of the form $2^\ell m$, where ℓ is a nonnegative integer and m is an odd integer that is not 1, 3, or 9.

Fici et al. [6] give the first few values of the sequence $\gamma(k)$ and speculate that the sequence grows linearly in k. We will prove that this is indeed the case. In fact, it is the aim of this paper to prove the following:

- $\displaystyle \frac{1}{2} \leq \liminf_{k\to\infty} \frac{\gamma(k)}{k} \leq \frac{9}{10}$
- $\displaystyle 1 \leq \limsup_{k\to\infty} \frac{\gamma(k)}{k} \leq \frac{3}{2}$
- $\displaystyle \liminf_{k\to\infty} \frac{\Gamma(k)}{k} = \frac{3}{2}$
- $\displaystyle \limsup_{k\to\infty} \frac{\Gamma(k)}{k} = 3.$

Despite these asymptotic results, there are many open problems arising from consideration of the sets $\mathcal{F}(k)$ (such as the cardinality of $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$) that we have not investigated; we discuss some of these problems at the end of the paper.

2. The Thue-Morse Word: Background and Notation

Our primary focus is on the Thue-Morse word $\mathbf{t}$. In this brief section, we discuss some of the basic properties of this word that we will need when proving our asymptotic results.

Let $\mathbf{t}_i$ denote the i^{th} letter of $\mathbf{t}$ so that $\mathbf{t} = \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\cdots$. The number $\mathbf{t}_i$ has the same parity as the number of 1's in the binary expansion of $i-1$. For any positive integers α, β with $\alpha \leq \beta$, define $\langle \alpha, \beta \rangle = \mathbf{t}_\alpha \mathbf{t}_{\alpha+1} \cdots \mathbf{t}_\beta$. In his seminal 1912 paper, Thue proved that $\mathbf{t}$ is overlap-free [7]. This means that if x and y are finite words and x is nonempty, then $xyxyx$ is not a factor of $\mathbf{t}$.

Equivalently, if a, b, n are positive integers satisfying $a < b \leq a + n$, then $\langle a, a+n \rangle \neq \langle b, b+n \rangle$. Note that this implies that $\mathbf{t}$ is cube-free.

We write $\mathbb{A}^{\leq\omega}$ to denote the set of all words over an alphabet $\mathbb{A}$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be sets of words. A *morphism* $f \colon \mathcal{W}_1 \to \mathcal{W}_2$ is a function satisfying $f(xy) = f(x)f(y)$ for all words $x, y \in \mathcal{W}_1$. A morphism is uniquely determined by where it sends letters. Let $\mu \colon \{0,1\}^{\leq\omega} \to \{01,10\}^{\leq\omega}$ denote the morphism defined by $\mu(0) = 01$ and $\mu(1) = 10$. Also, define a morphism $\sigma \colon \{01,10\}^{\leq\omega} \to \{0,1\}^{\leq\omega}$ by $\sigma(01) = 0$ and $\sigma(10) = 1$ so that $\sigma = \mu^{-1}$. The words $\mathbf{t}$ and $\overline{\mathbf{t}}$ are the unique one-sided infinite words over the alphabet $\{0,1\}$ that are fixed by μ. Because $\mu(\mathbf{t}) = \mathbf{t}$, we may view $\mathbf{t}$ as a word over the alphabet $\{01,10\}$. In particular, this means that $\mathbf{t}_{2i-1} \neq \mathbf{t}_{2i}$ for all positive integers i. In addition, if α and β are nonnegative integers with $\alpha < \beta$, then $\langle 2\alpha+1, 2\beta \rangle \in \{01,10\}^{\leq\omega}$. Recall the definitions of A_n and B_n from Definition 1.1. Observe that $A_n = \mu^n(0)$ and $B_n = \mu^n(1)$. Because $\mu^n(\mathbf{t}) = \mathbf{t}$, the Thue-Morse word is actually a word over the alphabet $\{A_n, B_n\}$. This leads us to the following simple but useful fact.

Fact 2.1. *For any positive integers n and r, $\langle 2^n r + 1, 2^n(r+1) \rangle = \mu^n(t_{r+1})$.*

3. Asymptotics for $\Gamma(k)$

In this section, we prove that $\liminf\limits_{k\to\infty} \Gamma(k)/k = 3/2$ and $\limsup\limits_{k\to\infty} \Gamma(k)/k = 3$. The following proposition will prove very useful when we do so.

Proposition 3.1. *Let $m \geq 2$ be an integer, and let $\delta(m) = \lceil \log_2(m/3) \rceil$.*

(i) If y and v are words such that yvy is a factor of $\mathbf{t}$ and $|y| = m$, then $2^{\delta(m)}$ divides $|yv|$.
(ii) There is a factor of $\mathbf{t}$ of the form yvy such that $|y| = m$ and $2^{\delta(m)+1}$ does not divide $|yv|$.

Proof. We first prove (ii) by induction on m. If $m = 2$, we may simply set $y = 01$ and $v = 1$. If $m = 3$, we may set $y = 101$ and $v = \varepsilon$ (the empty word). Now, assume $m \geq 4$. First, suppose m is even. By induction, we can find a factor of $\mathbf{t}$ of the form yvy such that $|y| = m/2$ and such that $2^{\delta(m/2)+1}$ does not divide $|yv|$. Note that $\mu(y)\mu(v)\mu(y)$ is a factor of $\mathbf{t}$ and that $2^{\delta(m/2)+2}$ does not divide $2|yv| = |\mu(y)\mu(v)|$. Since $\delta(m/2) + 2 = \delta(m) + 1$, we are done. Now, suppose m is odd. Because $m+1$ is even, we may use the above argument to find a factor $y'v'y'$ of $\mathbf{t}$ with $|y'| = m+1$ such that $2^{\delta(m+1)+1}$ does not divide $|y'v'|$. It is easy to show that $\delta(m) = \delta(m+1)$ because $m > 3$ is odd. This means that $2^{\delta(m)+1}$ does not divide $|y'v'|$. Let a be the last letter of y', and write $y' = y''a$. Put $v'' = av'$. Then $y''v''y''$ is a factor of $\mathbf{t}$ with $|y''| = m$ and $|y''v''| = |y'v'|$. This completes the inductive step.

We now prove (i) by induction on m. If $m \leq 3$, the proof is trivial because $\delta(2) = \delta(3) = 0$. Therefore, assume $m \geq 4$. Assume that yvy is a factor of $\mathbf{t}$ and $|y| = m$. Let us write $\mathbf{t} = xyvyz$.

Suppose by way of contradiction that $|vy|$ is odd. Then $|xy|$ and $|xyvy|$ have different parities. Write $y = y_1 a$, where a is the last letter of y. Either xy or $xyvy$ is an even-length prefix of $\mathbf{t}$, and is therefore a word in $\{01,10\}^{\leq\omega}$. It follows that the second-to-last letter of y is $\overline{a}$, so we may write $y_1 = y_2\overline{a}$. We now observe that one of the words xy_1 and $xyvy_1$ is an even-length prefix of $\mathbf{t}$, so the same reasoning as before tells us that the second-to-last letter in y_1 is a. Therefore, $y = y_3 a\overline{a}a$ for some word y_3. We can continue in this fashion to see that $a\overline{a}a\overline{a}a$ is a suffix of vy. This is impossible since $\mathbf{t}$ is overlap-free. Hence, $|vy|$ must be even. We now consider four cases corresponding to the possible parities of $|x|$ and m.

Case 1: $|x|$ and $|y| = m$ are both even. We just showed $|vy|$ is even, so all of the words $x, xy, xyv, xyvy$ are even-length prefixes of $\mathbf{t}$. This means that $x, y, v, z \in \{01, 10\}^{\leq\omega}$, so $\mathbf{t} = \sigma(x)\sigma(y)\sigma(v)\sigma(y)\sigma(z)$. By induction, we see that $2^{\delta(|\sigma(y)|)}$ divides $|\sigma(y)\sigma(v)|$. Because $\delta(|\sigma(y)|) = \delta(m/2) = \delta(m) - 1$ and $|\sigma(y)\sigma(v)| = |yv|/2$, it follows that $2^{\delta(m)}$ divides $|yv|$.

Case 2: $|x|$ is odd and m is even. As in the previous case, $|v|$ must be even. Let a, b, c be the last letters of y, v, x, respectively. Write $y = y_0 a$, $v = v_0 b$, $x = x_0 c$. We have $\mathbf{t} = x_0 c y_0 a v_0 b y_0 a z$. Note that $|x_0|$, $|cy_0|$, $|av_0|$, and $|by_0|$ are all even. In particular, cy_0 and by_0 are both in $\{01, 10\}^{\leq\omega}$. As a consequence, $b = c$. Setting $x' = x_0$, $y' = by_0$, $v' = av_0$, $z' = az$, we find that $\mathbf{t} = x'y'v'y'z'$. We are now in the same situation as in the previous case because $|x'|$ is even and $|y'| = m$. Consequently, $2^{\delta(m)}$ divides $|y'v'| = |yv|$.

Case 3: m is odd and $|x|$ is even. Let a be the last letter of y. Both v and z start with the letter $\overline{a}$, so we may write $v = \overline{a}v_1$ and $z = \overline{a}z_1$. Put $x_1 = x$ and $y_1 = y\overline{a}$. We have $\mathbf{t} = x_1 y_1 v_1 y_1 z_1$. Because $|x_1|$ and $|y_1| = m + 1$ are both even, we know from the first case that $2^{\delta(m+1)}$ divides $|y_1 v_1| = |yv|$. Now, simply observe that $\delta(m) = \delta(m + 1)$ because $m > 3$ is odd.

Case 4: m and $|x|$ are both odd. Let d be the first letter of y. Both x and v end in the letter $\overline{d}$, so we may write $x = x_2\overline{d}$ and $v = v_2\overline{d}$. Let $y_2 = \overline{d}y$ and $z_2 = z$. Then $\mathbf{t} = x_2 y_2 v_2 y_2 z_2$. Because $|x_2|$ and $|y_2| = m+1$ are both even, we know that $2^{\delta(m+1)}$ divides $|y_2 v_2| = |yv|$. Again, $\delta(m) = \delta(m+1)$. □

Corollary 3.1. *Let m be a positive integer, and let $\delta(m) = \lceil \log_2(m/3) \rceil$. If $k \geq 3$ and $m \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$, then $k - 1 \geq 2^{\delta(m)}$.*

Proof. There exist integers n_1 and n_2 with $0 \leq n_1 < n_2 \leq k - 1$ such that $\langle n_1 m + 1, (n_1 + 1)m \rangle = \langle n_2 m + 1, (n_2 + 1)m \rangle$. Let $y = \langle n_1 m + 1, (n_1 + 1)m \rangle$ and $v = \langle (n_1 + 1)m + 1, n_2 m \rangle$. The word yvy is a factor of $\mathbf{t}$, and $|y| = m$. According to Proposition 3.1, $2^{\delta(m)}$ divides $|yv| = (n_2 - n_1)m$, where $\delta(m) = \lceil \log_2(m/3) \rceil$. Since m is odd, $2^{\delta(m)}$ divides $n_2 - n_1$. This shows that $k - 1 \geq n_2 \geq n_2 - n_1 \geq 2^{\delta(m)}$. □

The following lemma is somewhat technical, but it will be useful for constructing specific pairs of identical factors of the Thue-Morse word. These specific pairs of factors will provide us with odd positive integers m for which $\mathfrak{K}(m)$ is relatively small. We will then make use of the fact, which follows immediately from Definitions 1.2 and 1.3, that $\Gamma(k) \geq m$ whenever $k \geq \mathfrak{K}(m)$.

Lemma 3.1. *Suppose r, m, ℓ, h, p, q are nonnegative integers satisfying the following conditions:*

- $h < 2^{\ell-2}$
- $rm = p \cdot 2^{\ell+1} + 2^{\ell-1} + h$
- $(r + 1)m \leq p \cdot 2^{\ell+1} + 5 \cdot 2^{\ell-2}$
- $(r + 2^{\ell-2})m = q \cdot 2^{\ell+1} + 3 \cdot 2^{\ell-2} + h$
- $\mathbf{t}_{p+1} \neq \mathbf{t}_{q+1}$.

Then $\langle rm + 1, (r + 1)m \rangle = \langle (r + 2^{\ell-2})m + 1, (r + 2^{\ell-2} + 1)m \rangle$, and $\mathfrak{K}(m) \leq r + 2^{\ell-2} + 1$.

Proof. Let $u = \langle rm + 1, (r + 1)m \rangle$ and $v = \langle (r + 2^{\ell-2})m + 1, (r + 2^{\ell-2} + 1)m \rangle$. Let us assume $\mathbf{t}_{p+1} = 0$; a similar argument holds if we assume instead that $\mathbf{t}_{p+1} = 1$. According to Fact 2.1,

$$\langle p \cdot 2^{\ell+1} + 1, (p + 1)2^{\ell+1} \rangle = A_{\ell+1} = A_{\ell-2}B_{\ell-2}B_{\ell-2}A_{\ell-2}B_{\ell-2}A_{\ell-2}A_{\ell-2}B_{\ell-2}.$$

<table>
<tr><td colspan="8">$A_{\ell+1}$</td><td></td><td colspan="8">$B_{\ell+1}$</td></tr>
<tr><td>$A_{\ell-2}$</td><td>$B_{\ell-2}$</td><td>$B_{\ell-2}$</td><td>$A_{\ell-2}$</td><td>$B_{\ell-2}$</td><td>$A_{\ell-2}$</td><td>$A_{\ell-2}$</td><td>$B_{\ell-2}$</td><td></td><td>$B_{\ell-2}$</td><td>$A_{\ell-2}$</td><td>$A_{\ell-2}$</td><td>$B_{\ell-2}$</td><td>$A_{\ell-2}$</td><td>$B_{\ell-2}$</td><td>$B_{\ell-2}$</td><td>$A_{\ell-2}$</td></tr>
<tr><td colspan="2"></td><td>x</td><td>u</td><td>y</td><td colspan="7"></td><td>x'</td><td>v</td><td>y'</td><td colspan="2"></td></tr>
</table>

FIGURE 1. An illustration of the proof of Lemma 3.1.

We may now use the first three conditions to see that $B_{\ell-2}A_{\ell-2}B_{\ell-2} = xuy$ for some words x and y such that $|x| = h$ and $|y| = p \cdot 2^{\ell+1} + 5 \cdot 2^{\ell-2} - (r+1)m$ (see Figure 1).

We know from the last condition that $\mathbf{t}_{q+1} = 1$, so

$$\langle q \cdot 2^{\ell+1} + 1, (q+1)2^{\ell+1} \rangle = B_{\ell+1} = B_{\ell-2}A_{\ell-2}A_{\ell-2}B_{\ell-2}A_{\ell-2}B_{\ell-2}B_{\ell-2}A_{\ell-2}.$$

The fourth condition tells us that $B_{\ell-2}A_{\ell-2}B_{\ell-2} = x'vy'$ for some words x' and y' with $|x'| = h$. We have shown that $xuy = x'vy'$, where $|x| = |x'|$ and $|u| = |v|$. Hence, $u = v$. It follows that the prefix of $\mathbf{t}$ of length $(r + 2^{\ell-2} + 1)m$ is not a $(r + 2^{\ell-2} + 1)$-anti-power, so $\mathfrak{K}(m) \leq r + 2^{\ell-2} + 1$ by definition. □

We may now use Lemma 3.1 and Proposition 3.1 to prove that $\limsup_{k\to\infty} \Gamma(k)/k = 3$. Recall that if $k \geq 3$, then $\Gamma(k) \geq 3$ because $3 \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$. A particular consequence of the following theorem is that $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is finite. It follows that if $k \geq 3$, then $\Gamma(k)$ is an odd positive integer.

Theorem 3.1. *Let $\Gamma(k)$ be as in Definition 1.2. For all integers $k \geq 3$, we have $\Gamma(k) \leq 3k - 4$. Furthermore,* $\limsup_{k\to\infty} \dfrac{\Gamma(k)}{k} = 3$.

Proof. Fix $k \geq 3$, and let $m \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$. If $m \leq 5$, then $m \leq 3k - 4$ as desired, so assume $m \geq 7$. By Corollary 3.1, $k - 1 \geq 2^{\delta(m)}$, where $\delta(m) = \lceil \log_2(m/3) \rceil$. Since $m \geq 7$ is odd, $\delta(m) > \log_2(m/3)$. This shows that $k - 1 \geq 2^{\delta(m)} > m/3$, so $m \leq 3k - 4$. Consequently, $\Gamma(k) \leq 3k - 4$.

We now show that $\limsup_{k\to\infty} \dfrac{\Gamma(k)}{k} = 3$. For each positive integer α, let $k_\alpha = 2^{2\alpha} + 2^\alpha + 2$. Let us fix an integer $\alpha \geq 3$ and set $r = 2^\alpha + 1$, $m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1$, $\ell = 2\alpha + 2$, $h = 1$, $p = 3 \cdot 2^{\alpha-3}$, and $q = 3 \cdot 2^{2\alpha-3} + 2^{\alpha-2}$. One may easily verify that these values of r, m, ℓ, h, p, and q satisfy the first four of the five conditions listed in Lemma 3.1. Recall that the parity of $\mathbf{t}_i$ is the same as the parity of the number of 1's in the binary expansion of $i - 1$. The binary expansion of p has exactly two 1's, and the binary expansion of q has exactly three 1's. Therefore, $\mathbf{t}_{p+1} = 0 \neq 1 = \mathbf{t}_{q+1}$. This shows that all of the conditions in Lemma 3.1 are satisfied, so $\mathfrak{K}(m) \leq r + 2^{\ell-2} + 1 = k_\alpha$. The prefix of $\mathbf{t}$ of length $k_\alpha m$ is not a k_α-anti-power, so $\Gamma(k_\alpha) \geq m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1$. For each $\alpha \geq 3$,

$$\frac{\Gamma(k_\alpha)}{k_\alpha} \geq \frac{3 \cdot 2^{2\alpha} - 2^\alpha + 1}{2^{2\alpha} + 2^\alpha + 2}.$$ □

In the preceding proof, we found an increasing sequence of positive integers $(k_\alpha)_{\alpha \geq 3}$ with the property that $\Gamma(k_\alpha)/k_\alpha \to 3$ as $\alpha \to \infty$. It will be useful to have two other sequences with similar properties. This is the content of the following lemma.

Lemma 3.2. *For integers $\alpha \geq 3$, $\beta \geq 9$, and $\rho \geq 4$, define*

$$k_\alpha = 2^{2\alpha} + 2^\alpha + 2, \quad K_\beta = 2^{2\beta+1} + 3 \cdot 2^{\beta+3} + 49, \quad \text{and} \quad \kappa_\rho = 2^\rho + 2.$$

We have

$$\Gamma(k_\alpha) \geq 3 \cdot 2^{2\alpha} - 2^\alpha + 1, \quad \Gamma(K_\beta) \geq 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1, \quad \text{and} \quad \Gamma(\kappa_\rho) \geq 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1,$$

where $\chi(\rho) = \begin{cases} 1, & \text{if } \rho \equiv 0 \pmod 2; \\ 2, & \text{if } \rho \equiv 1 \pmod 2. \end{cases}$

Proof. We already derived the lower bound for $\Gamma(k_\alpha)$ in the proof of Theorem 3.1. To prove the lower bound for $\Gamma(K_\beta)$, put $r = 3 \cdot 2^{\beta+3} + 48$, $m = 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$, $\ell = 2\beta + 3$, $h = 48$, $p = 9 \cdot 2^\beta + 17$, and $q = 3 \cdot 2^{2\beta-2} + 143 \cdot 2^{\beta-4} + 17$. Straightforward calculations show that these choices of r, m, ℓ, h, p, and q satisfy the first four conditions of Lemma 3.1. The binary expansion of p has exactly four 1's while that of q has exactly nine 1's (it is here that we require $\beta \geq 9$). It follows that $\mathbf{t}_{p+1} = 0 \neq 1 = \mathbf{t}_{q+1}$, so the final condition in Lemma 3.1 is also satisfied. The lemma tells us that $\mathfrak{K}(m) \leq r + 2^{\ell-2} + 1 = K_\beta$, so the prefix of $\mathbf{t}$ of length $K_\beta m$ is not a K_β-anti-power. Hence, $\Gamma(K_\beta) \geq m = 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$.

To prove the lower bound for κ_ρ, we again invoke Lemma 3.1. Let $r' = 1$, $m' = 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1$, $\ell' = \rho + 2$, $h' = 2^{\rho-1} - 8\chi(\rho) + 1$, $p' = 0$, and $q' = 5 \cdot 2^{\rho-4} - \chi(\rho)$. These choices satisfy the first four conditions in Lemma 3.1. The binary expansion of q' has an odd number of 1's, so $\mathbf{t}_{p'+1} = \mathbf{t}_1 = 0 \neq 1 = \mathbf{t}_{q'+1}$. We now know that $\mathfrak{K}(m') \leq r' + 2^{\ell'-2} + 1 = \kappa_\rho$, so $\Gamma(\kappa_\rho) \geq m' = 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1$. □

We now use the sequences $(k_\alpha)_{\alpha \geq 3}$, $(K_\beta)_{\beta \geq 9}$, and $(\kappa_\rho)_{\rho \geq 4}$ to prove that $\liminf_{k\to\infty} (\Gamma(k)/k) = 3/2$.

Theorem 3.2. *Let $\Gamma(k)$ be as in Definition 1.2. We have* $\liminf_{k\to\infty} \dfrac{\Gamma(k)}{k} = \dfrac{3}{2}$.

Proof. Let $k \geq 3$ be a positive integer, and let $m = \Gamma(k)$. Put $\delta(m) = \lceil \log_2(m/3) \rceil$. Corollary 3.1 tells us that $k - 1 \geq 2^{\delta(m)}$. Suppose k is a power of 2, say $k = 2^\lambda$. Then the inequality $k - 1 \geq 2^{\delta(m)}$ forces $\delta(m) \leq \lambda - 1$. Thus, $m \leq 3 \cdot 2^{\lambda-1} = \dfrac{3}{2}k$. This shows that $\dfrac{\Gamma(k)}{k} \leq \dfrac{3}{2}$ whenever k is a power of 2, so $\liminf_{k\to\infty} \dfrac{\Gamma(k)}{k} \leq \dfrac{3}{2}$.

To prove the reverse inequality, we will make use of Lemma 3.2. Recall the definitions of k_α, K_β, κ_ρ, and $\chi(\rho)$ from that lemma. Fix $k \geq \kappa_{18}$, and put $m = \Gamma(k)$. Because $k \geq \kappa_{18}$, we may use Lemma 3.2 and the fact that Γ is nondecreasing (see Remark 1.1) to see that $m = \Gamma(k) \geq \Gamma(\kappa_{18}) \geq 5 \cdot 2^{17} - 7$. Let $\ell = \lceil \log_2 m \rceil$ so that $2^{\ell-1} < m < 2^\ell$. Note that $\ell \geq 20$. Let us first assume that $3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2} < m < 2^\ell$. Lemma 3.2 tells us that $\Gamma(\kappa_{\ell-1}) \geq 5 \cdot 2^{\ell-2} - 8\chi(\ell-1) + 1$. We also know that $5 \cdot 2^{\ell-2} - 8\chi(\ell-1) + 1 > m$, so $\Gamma(\kappa_{\ell-1}) > m$. Because Γ is nondecreasing, $\kappa_{\ell-1} > k$. Thus,

$$\frac{\Gamma(k)}{k} > \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{\kappa_{\ell-1}} = \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{2^{\ell-1} + 2} \tag{1}$$

if $3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2} < m < 2^\ell$.

Next, assume $2^{\ell-1} < m \leq 3\cdot 2^{\ell-2} - 2^{(\ell-2)/2}$ and ℓ is even. According to Lemma 3.2, $\Gamma(k_{(\ell-2)/2}) \geq 3\cdot 2^{\ell-2} - 2^{(\ell-2)/2} + 1 > m$. Because Γ is nondecreasing, $k < k_{(\ell-2)/2}$. Therefore,

$$\frac{\Gamma(k)}{k} > \frac{2^{\ell-1}}{k_{(\ell-2)/2}} = \frac{2^{\ell-1}}{2^{\ell-2} + 2^{(\ell-2)/2} + 2}. \tag{2}$$

Finally, suppose $2^{\ell-1} < m \leq 3\cdot 2^{\ell-2} - 2^{(\ell-2)/2}$ and ℓ is odd. Lemma 3.2 states that $\Gamma(K_{(\ell-3)/2}) \geq 3\cdot 2^{\ell-2} - 2^{(\ell-5)/2} + 1 > m$. We know that $k < K_{(\ell-3)/2}$ because Γ is nondecreasing. As a consequence,

$$\frac{\Gamma(k)}{k} > \frac{2^{\ell-1}}{K_{(\ell-3)/2}} = \frac{2^{\ell-1}}{2^{\ell-2} + 3\cdot 2^{(\ell+3)/2} + 49}. \tag{3}$$

The inequalities in (1), (2), and (3) show that in all cases, $\dfrac{\Gamma(k)}{k} > \dfrac{3\cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{2^{\ell-1}+2}$. Because $\ell \to \infty$ as $k \to \infty$ ($\Gamma(k)$ cannot be bounded since we have just shown $\Gamma(k)/k$ is bounded away from 0), we find that $\liminf_{k\to\infty} \Gamma(k)/k \geq 3/2$. □

4. Asymptotics for $\gamma(k)$

Having demonstrated that $\liminf_{k\to\infty}(\Gamma(k)/k) = 3/2$ and $\limsup_{k\to\infty}(\Gamma(k)/k) = 3$, we turn our attention to $\gamma(k)$. To begin the analysis, we prove some lemmas that culminate in an upper bound for $\mathfrak{K}(m)$ for any odd positive integer m. It will be useful to keep in mind that if j is a nonnegative integer, then $\mathbf{t}_{2j} \neq \mathbf{t}_{2j+1} = \mathbf{t}_{j+1}$ and $\mathbf{t}_{4j+2} = \mathbf{t}_{4j+3}$.

Lemma 4.1. *Let m be an odd positive integer, and let $\ell = \lceil \log_2 m \rceil$. If $\mathfrak{K}(m) > 2^\ell + 1$, then $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$ and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$.*

Proof. Let $w_0 = \langle 1, m\rangle$, $w_1 = \langle 2^{\ell-1}m + 1, (2^{\ell-1}+1)m\rangle$, and $w_2 = \langle 2^\ell m + 1, (2^\ell + 1)m\rangle$. The words w_0, w_1, w_2 must be distinct because $\mathfrak{K}(m) > 2^\ell + 1$. For each $n \in \{0,1,2\}$, w_n is a prefix of $\langle nm2^{\ell-1} + 1, (nm+2)2^{\ell-1}\rangle = \mu^{\ell-1}(\mathbf{t}_{nm+1}\mathbf{t}_{nm+2})$. It follows that $\mathbf{t}_1\mathbf{t}_2$, $\mathbf{t}_{m+1}\mathbf{t}_{m+2}$, and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2}$ are distinct. Since $\mathbf{t}_1\mathbf{t}_2 = 01$ and $\mathbf{t}_{2m+1} \neq \mathbf{t}_{2m+2}$, we must have $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$. Now, $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = \mu(\mathbf{t}_{m+1})$, so $\mathbf{t}_{m+1} = 1$. This forces $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$. □

Lemma 4.2. *Let $m \geq 3$ be an odd integer, and let $\ell = \lceil \log_2 m \rceil$. Suppose there is a positive integer j such that $\mathbf{t}_j\mathbf{t}_{j+1} = \mathbf{t}_{m+j}\mathbf{t}_{m+j+1}$. Then $\mathfrak{K}(m) < \left(1 + \dfrac{j+1}{m}\right)2^\ell$.*

Proof. First, observe that

$$\langle 2^\ell(j-1)+1, 2^\ell(j+1)\rangle = \mu^\ell(\mathbf{t}_j\mathbf{t}_{j+1}) = \mu^\ell(\mathbf{t}_{m+j}\mathbf{t}_{m+j+1}) = \langle 2^\ell(m+j-1)+1, 2^\ell(m+j+1)\rangle. \tag{4}$$

Because $|\langle 2^\ell(j-1)+1, 2^\ell(j+1)\rangle| = 2^{\ell+1} > 2m$, there is a nonnegative integer r such that

$$\langle 2^\ell(j-1)+1, 2^\ell(j+1)\rangle = w\langle rm+1, (r+1)m\rangle z \tag{5}$$

for some nonempty words w and z. Note that $r+1 < \dfrac{2^\ell(j+1)}{m}$. It follows from (5) that

$$2^\ell(m+j-1)+1 < 2^\ell m + rm + 1 < 2^\ell m + (r+1)m < 2^\ell(m+j+1),$$

so

$$\langle 2^\ell(m+j-1)+1, 2^\ell(m+j+1)\rangle = w'\langle (2^\ell+r)m+1, (2^\ell+r+1)m\rangle z'$$

for some nonempty words w' and z'. Note that $|w'| = (2^\ell+r)m-2^\ell(m+j-1) = rm-2^\ell(j-1) = |w|$. Combining this fact with (4), we find that

$$\langle rm+1, (r+1)m\rangle = \langle (2^\ell+r)m+1, (2^\ell+r+1)m\rangle.$$

Consequently,

$$\mathfrak{K}(m) \le 2^\ell + r + 1 < 2^\ell + \frac{2^\ell(j+1)}{m}.$$

□

Lemma 4.3. *Let m be an odd positive integer with $m \not\equiv 1 \pmod 8$, and let $\ell = \lceil \log_2 m\rceil$. We have $\mathfrak{K}(m) < \left(1+\frac{37}{m}\right)2^\ell$.*

Proof. Suppose instead that $\mathfrak{K}(m) \ge \left(1+\frac{37}{m}\right)2^\ell$. Let us assume for the moment that $m \not\equiv 29 \pmod{32}$. We will obtain a contradiction to Lemma 4.2 by exhibiting a positive integer $j \le 36$ such that $\mathbf{t}_j\mathbf{t}_{j+1} = \mathbf{t}_{m+j}\mathbf{t}_{m+j+1}$. Because $\mathfrak{K}(m) > 2^\ell+1$, Lemma 4.1 tells us that $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$ and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$.

First, assume $m \equiv 3 \pmod 4$. We have $\langle m+2, m+5\rangle = \mu^2(\mathbf{t}_{(m+5)/4})$, so either $\langle m+2, m+5\rangle = 0110$ or $\langle m+2, m+5\rangle = 1001$. Since $\mathbf{t}_{m+2} = 1$, we must have $\langle m+2, m+5\rangle = 1001$. This shows that $\mathbf{t}_{m+4}\mathbf{t}_{m+5} = 01 = \mathbf{t}_4\mathbf{t}_5$, so we may set $j = 4$.

Next, assume $m \equiv 5 \pmod 8$. Let $x01^s01$ be the binary expansion of m, where x is some (possibly empty) string of 0's and 1's. As $m \equiv 5 \pmod 8$ and $m \not\equiv 29 \pmod{32}$, we must have $1 \le s \le 2$. Because $\mathbf{t}_{m+1} = 1$, the number of 1's in the binary expansion of m is odd. This means that the parity of the number of 1's in x is the same as the parity of s.

Suppose $s = 1$. The binary expansion of $m+3$ is the string $x1000$, which contains an even number of 1's. As a consequence, $\mathbf{t}_{m+4} = 0$. The binary expansion of $m+4$ is $x1001$, so $\mathbf{t}_{m+5} = 1$. This shows that $\mathbf{t}_{m+4}\mathbf{t}_{m+5} = 01 = \mathbf{t}_4\mathbf{t}_5$, so we may set $j = 4$.

Suppose that $s = 2$ and that x ends in a 0, say $x = y0$. Note that y contains an even number of 1's. The binary expansions of $m+19$ and $m+20$ are $y100000$ and $y100001$, respectively, so $\mathbf{t}_{m+20}\mathbf{t}_{m+21} = 10 = \mathbf{t}_{20}\mathbf{t}_{21}$. We may set $j = 20$ in this case.

Assume now that $s = 2$ and that x ends in a 1. Let us write $x = x'01^{s'}$, where x' is a (possibly empty) binary string. For this last step, we may need to add additional 0's to the beginning of x. Doing so does not raise any issues because it does not change the number of 1's in x. The binary expansion of m is $x'01^{s'}01101$. Note that the parity of the number of 1's in x' is the same as the parity of s'. The binary expansions of $m+19$ and $m+35$ are $x'10^{s'+5}$ and $x'10^{s'}10000$, respectively. If s' is even, then we may put $j = 20$ because $\mathbf{t}_{m+20}\mathbf{t}_{m+21} = 10 = \mathbf{t}_{20}\mathbf{t}_{21}$. If s' is odd, then we may set $j = 36$ because $\mathbf{t}_{m+36}\mathbf{t}_{m+37} = 10 = \mathbf{t}_{36}\mathbf{t}_{37}$.

We now handle the case in which $m \equiv 29 \pmod{32}$. Say $m = 32n-3$. Let b be the number of 1's in the binary expansion of n. The binary expansion of $m+17 = 32n+14$ has $b+3$ 1's. Similarly, the binary expansions of $m+18$, $m+19$, $2m+17$, $2m+18$, and $2m+19$ have $b+4$, $b+1$, $b+3$, $b+2$, and $b+3$ 1's, respectively. This means that $\mathbf{t}_{m+18}\mathbf{t}_{m+19}\mathbf{t}_{m+20} = \mathbf{t}_{2m+18}\mathbf{t}_{2m+19}\mathbf{t}_{2m+20}$. Therefore,

$$\langle (m+17)2^{\ell-1}+1, (m+20)2^{\ell-1}\rangle = \mu^{\ell-1}(\mathbf{t}_{m+18}\mathbf{t}_{m+19}\mathbf{t}_{m+20})$$

$$= \mu^{\ell-1}(\mathbf{t}_{2m+18}\mathbf{t}_{2m+19}\mathbf{t}_{2m+20}) = \langle (2m+17)2^{\ell-1}+1, (2m+20)2^{\ell-1}\rangle. \tag{6}$$

We have $\bigcup_{r=9}^{17}\left(\frac{17}{2r},\frac{10}{r+1}\right)=\left(\frac{1}{2},1\right)$, so there exists some $r\in\{9,10,\ldots,17\}$ such that $\frac{17}{2r}<\frac{m}{2^\ell}<\frac{10}{r+1}$. Equivalently, $17\cdot 2^{\ell-1}<rm<(r+1)m<20\cdot 2^{\ell-1}$. It follows that there are nonempty words w and z such that $\langle (m+17)2^{\ell-1}+1,(m+20)2^{\ell-1}\rangle = w\langle (r+2^{\ell-1})m+1,(r+2^{\ell-1}+1)m\rangle z$. Similarly, there are nonempty words w' and z' such that $\langle (2m+17)2^{\ell-1}+1,(2m+20)2^{\ell-1}\rangle = w'\langle (r+2^\ell)m+1,(r+2^\ell+1)m\rangle z'$. Note that $|w|=rm-17\cdot 2^{\ell-1}=|w'|$. Invoking (6) yields $\langle (r+2^{\ell-1})m+1,(r+2^{\ell-1}+1)m\rangle=\langle (r+2^\ell)m+1,(r+2^\ell+1)m\rangle$. This shows that $\mathfrak{K}(m)\le r+2^\ell+1\le 2^\ell+18$, securing our final contradiction to the assumption that $\mathfrak{K}(m)\ge\left(1+\frac{37}{m}\right)2^\ell$. $\square$

Lemma 4.4. *Let m be an odd positive integer, and let $\ell=\lceil\log_2 m\rceil$. Suppose $m=2^Lh+1$, where L and h are integers with $L\ge 3$ and h odd. We have $\mathfrak{K}(m)<\left(1+\frac{2^{L+1}+4}{m}\right)2^\ell$.*

Proof. Suppose instead that $\mathfrak{K}(m)\ge\left(1+\frac{2^{L+1}+4}{m}\right)2^\ell$. We will obtain a contradiction to Lemma 4.2 by finding a positive integer $j\le 2^{L+1}+3$ satisfying $\mathbf{t}_j\mathbf{t}_{j+1}=\mathbf{t}_{m+j}\mathbf{t}_{m+j+1}$. Let $x01^s0^{L-1}1$ be the binary expansion of m, and note that $s\ge 1$. Let N be the number of 1's in x. The binary expansions of $m+2^L+2$, $m+2^L+3$, $m+2^{L+1}+2$, and $m+2^{L+1}+3$ are $x10^{s+L-2}11$, $x10^{s+L-3}100$, $x10^{s-1}10^{L-2}11$, and $x10^{s-1}10^{L-3}100$. This shows that $\mathbf{t}_{m+2^L+3}\mathbf{t}_{m+2^L+4}=10$ if N is even and $\mathbf{t}_{m+2^{L+1}+3}\mathbf{t}_{m+2^{L+1}+4}=10$ if N is odd. Observe that $\mathbf{t}_{2^L+3}\mathbf{t}_{2^L+4}=\mathbf{t}_{2^{L+1}+3}\mathbf{t}_{2^{L+1}+4}=10$. Therefore, we may put $j=2^L+3$ if N is even and $j=2^{L+1}+3$ if N is odd. $\square$

Lemma 4.5. *Let m be an odd positive integer, and let $\ell=\lceil\log_2 m\rceil$. Assume $m=2^Lh+1$ for some integers L and h with $L\ge 3$ and h odd. If n is an integer such that $2\le n\le 2^{L-1}$, $\mathbf{t}_{m-n}=\mathbf{t}_{m-n+1}$, and $m\le\left(1-\frac{1}{2n+2}\right)2^\ell$, then $\mathfrak{K}(m)\le 2^\ell-n$.*

Proof. Let y and z be the binary expansions of $2^{L-1}-n$ and $2^{L-1}-n+1$, respectively. If necessary, let the strings y and z begin with additional 0's so that $|y|=|z|=L-1$. Let $x10^L$ be the binary expansion of $m-1$. The binary expansions of $m-2n-1$ and $2m-2n-1$ are $x0y0$ and $x01y1$, respectively. The quantities of 1's in these strings are of the same parity, so $\mathbf{t}_{m-2n}=\mathbf{t}_{2m-2n}$. Similarly, $\mathbf{t}_{m-2n+2}=\mathbf{t}_{2m-2n+2}$ because the binary expansions of $m-2n+1$ and $2m-2n+1$ are $x0z0$ and $x01z1$, respectively. Let $a=\mathbf{t}_{m-n}$. Because $\mathbf{t}_{m-n}=\mathbf{t}_{m-n+1}$ by hypothesis, we have $\mathbf{t}_{2m-2n}=\mathbf{t}_{2m-2n+2}=\overline{a}$. Therefore, $\mathbf{t}_{m-2n}=\mathbf{t}_{m-2n+2}=\overline{a}$. The word $\mathbf{t}$ is cube-free, so $\mathbf{t}_{m-2n}\mathbf{t}_{m-2n+1}\mathbf{t}_{m-2n+2}=\overline{a}a\overline{a}=\mathbf{t}_{2m-2n}\mathbf{t}_{2m-2n+1}\mathbf{t}_{2m-2n+2}$. Hence,

$$\langle (m-2n-1)2^{\ell-1}+1,(m-2n+2)2^{\ell-1}\rangle=\mu^{\ell-1}(\mathbf{t}_{m-2n}\mathbf{t}_{m-2n+1}\mathbf{t}_{m-2n+2})$$

$$=\mu^{\ell-1}(\mathbf{t}_{2m-2n}\mathbf{t}_{2m-2n+1}\mathbf{t}_{2m-2n+2})=\langle (2m-2n-1)2^{\ell-1}+1,(2m-2n+2)2^{\ell-1}\rangle. \tag{7}$$

Now, $m\in\left(2^{\ell-1},\left(1-\frac{1}{2n+2}\right)2^\ell\right]\subseteq\bigcup_{r=n}^{2n-1}\left[\frac{2n-2}{r}2^{\ell-1},\frac{2n+1}{r+1}2^{\ell-1}\right]$, so there is some $r\in\{n,n+1,\ldots,2n-1\}$ such that $\frac{2n-2}{r}2^{\ell-1}\le m\le\frac{2n+1}{r+1}2^{\ell-1}$. Equivalently, $(m-2n-1)2^{\ell-1}\le(2^{\ell-1}-r-1)m<(2^{\ell-1}-r)m\le(m-2n+2)2^{\ell-1}$. We find that

$$\langle (m-2n-1)2^{\ell-1}+1,(m-2n+2)2^{\ell-1}\rangle=w\langle (2^{\ell-1}-r-1)m+1,(2^{\ell-1}-r)m\rangle z$$

and

$$\langle (2m-2n-1)2^{\ell-1}+1,(2m-2n+2)2^{\ell-1}\rangle=w'\langle (2^\ell-r-1)m+1,(2^\ell-r)m\rangle z'$$

for some words w, w', z, z'. Because $|w| = (2n+1)2^{\ell-1} - (r+1)m = |w'|$, we may use (7) to deduce that

$$\langle (2^{\ell-1} - r - 1)m + 1, (2^{\ell-1} - r)m \rangle = \langle (2^{\ell} - r - 1)m + 1, (2^{\ell} - r)m \rangle.$$

This shows that $\mathfrak{K}(m) \le 2^{\ell} - r \le 2^{\ell} - n$ as desired. □

Lemma 4.6. *If m is an odd positive integer and $\ell = \lceil \log_2 m \rceil$, then $\mathfrak{K}(m) < 2^{\ell} + 2^{(\ell+5)/2} + 10$.*

Proof. We will assume that $m \ge 65$ (so $\ell \ge 7$). One may easily use a computer to check that the desired result holds when $m < 65$.

If $m \not\equiv 1 \pmod 8$, then Lemma 4.3 tells us that

$$\mathfrak{K}(m) < \left(1 + \frac{37}{m}\right) 2^{\ell} < 2^{\ell} + 74 \le 2^{\ell} + 2^{(\ell+5)/2} + 10.$$

Suppose that $m \equiv 1 \pmod 8$, and let $m = 2^L h + 1$, where $L \ge 3$ and h is odd. First, assume $m > \left(1 - \frac{1}{2^L - 4}\right) 2^{\ell}$. Because $2^L | 2^{\ell} - m + 1$ and $2^{\ell} - m + 1 > 0$, we have $2^L \le 2^{\ell} - m + 1 < \frac{2^{\ell}}{2^L - 4} + 1$. This implies that $2^{2L} - 4 \cdot 2^L < 2^{\ell} + 2^L - 4$, so $2^L < 2^{\ell - L} + 5 - 4 \cdot 2^{-L} < 2^{\ell - L + 2}$. Hence, $L \le \frac{\ell + 1}{2}$. By Lemma 4.4,

$$\mathfrak{K}(m) < \left(1 + \frac{2^{L+1} + 4}{m}\right) 2^{\ell} < 2^{\ell} + 2^{L+2} + 8 < 2^{\ell} + 2^{(\ell+5)/2} + 10.$$

Next, assume $m \le \left(1 - \frac{1}{2^L - 4}\right) 2^{\ell}$ and $L \ge 4$. Let n be the largest integer satisfying $m - n \equiv 2 \pmod 4$ and $n \le 2^{L-1}$. Note that $m \le \left(1 - \frac{1}{2n+2}\right) 2^{\ell}$ because $n \ge 2^{L-1} - 3$. As $m - n \equiv 2 \pmod 4$, we have $\mathbf{t}_{m-n} = \mathbf{t}_{m-n+1}$. We have shown that n satisfies the criteria specified in Lemma 4.5, so $\mathfrak{K}(m) \le 2^{\ell} - n < 2^{\ell} + 2^{(\ell+5)/2} + 10$.

Finally, if $L = 3$, then Lemma 4.4 tells us that

$$\mathfrak{K}(m) < \left(1 + \frac{20}{m}\right) 2^{\ell} < 2^{\ell} + 40 < 2^{\ell} + 2^{(\ell+5)/2} + 10.$$ □

At last, we are in a position to prove lower bounds for $\liminf_{k \to \infty} (\gamma(k)/k)$ and $\limsup_{k \to \infty} (\gamma(k)/k)$.

Theorem 4.1. *Let $\gamma(k)$ be as in Definition 1.2. We have*

$$\liminf_{k \to \infty} \frac{\gamma(k)}{k} \ge \frac{1}{2} \quad \textit{and} \quad \limsup_{k \to \infty} \frac{\gamma(k)}{k} \ge 1.$$

Proof. For each positive integer ℓ, let $g(\ell) = \lfloor 2^{\ell} + 2^{(\ell+5)/2} + 10 \rfloor + 1$. Lemma 4.6 implies that $\mathfrak{K}(m) < g(\ell)$ for all odd positive integers $m < 2^{\ell}$. It follows from the definition of γ that $\gamma(g(\ell)) \ge 2^{\ell} + 1$. Therefore,

$$\limsup_{k \to \infty} \frac{\gamma(k)}{k} \ge \limsup_{\ell \to \infty} \frac{\gamma(g(\ell))}{g(\ell)} \ge \lim_{\ell \to \infty} \frac{2^{\ell} + 1}{2^{\ell} + 2^{(\ell+5)/2} + 11} = 1.$$

Now, choose an arbitrary positive integer k, and let $\ell = \lceil \log_2(\gamma(k)) \rceil$. By the definition of γ, $k < \mathfrak{K}(\gamma(k))$. We may use Lemma 4.6 to find that

$$\frac{\gamma(k)}{k} > \frac{\gamma(k)}{2^\ell + 2^{(\ell+5)/2} + 10} > \frac{2^{\ell-1}}{2^\ell + 2^{(\ell+5)/2} + 10}.$$

Note that this implies that $\gamma(k) \to \infty$ as $k \to \infty$. It follows that $\ell \to \infty$ as $k \to \infty$, so

$$\liminf_{k\to\infty} \frac{\gamma(k)}{k} \geq \lim_{\ell\to\infty} \frac{2^{\ell-1}}{2^\ell + 2^{(\ell+5)/2} + 10} = \frac{1}{2}.$$

□

In our final theorem, we provide upper bounds for $\liminf\limits_{k\to\infty}(\gamma(k)/k)$ and $\limsup\limits_{k\to\infty}(\gamma(k)/k)$. This will complete our proof of all the asymptotic results mentioned in the introduction. Before proving this theorem, we need one lemma. In what follows, recall that the Thue-Morse word $\mathbf{t}$ is overlap-free. This means that if a, b, n are positive integers satisfying $a < b \leq a + n$, then $\langle a, a+n \rangle \neq \langle b, b+n \rangle$.

Lemma 4.7. *For each integer $\ell \geq 3$, we have*

$$\mathfrak{K}(3 \cdot 2^{\ell-2} + 1) > \frac{5 \cdot 2^{2\ell-3}}{3 \cdot 2^{\ell-2} + 1} \quad \text{and} \quad \mathfrak{K}(2^{\ell-1} + 3) > \frac{2^{2\ell-2}}{2^{\ell-1} + 3}.$$

Proof. Fix $\ell \geq 3$, and let $m = 3 \cdot 2^{\ell-2} + 1$ and $m' = 2^{\ell-1} + 3$. By the definitions of $\mathfrak{K}(m)$ and $\mathfrak{K}(m')$, there are nonnegative integers $r < \mathfrak{K}(m) - 1$ and $r' < \mathfrak{K}(m') - 1$ such that $\langle rm + 1, (r+1)m \rangle = \langle (\mathfrak{K}(m) - 1)m + 1, \mathfrak{K}(m)m \rangle$ and $\langle r'm' + 1, (r'+1)m' \rangle = \langle (\mathfrak{K}(m') - 1)m' + 1, \mathfrak{K}(m')m' \rangle$. According to Proposition 3.1, $2^{\ell-1}$ divides $(\mathfrak{K}(m) - 1)m - rm$ and $2^{\ell-2}$ divides $(\mathfrak{K}(m') - 1)m' - r'm'$. Since m and m' are odd, we know that $2^{\ell-1}$ divides $\mathfrak{K}(m) - r - 1$ and $2^{\ell-2}$ divides $\mathfrak{K}(m') - r' - 1$. If $\mathfrak{K}(m) - r - 1 \geq 2^\ell$, then $\mathfrak{K}(m) > \dfrac{5 \cdot 2^{2\ell-3}}{3 \cdot 2^{\ell-2} + 1}$ as desired. Therefore, we may assume $\mathfrak{K}(m) = r + 2^{\ell-1} + 1$. By the same token, we may assume that $\mathfrak{K}(m') = r' + 2^{\ell-2} + 1$.

With the aim of finding a contradiction, let us assume $\mathfrak{K}(m) \leq \dfrac{5 \cdot 2^{2\ell-3}}{m}$. Put

$$u = \langle rm + 1, (r+1)m \rangle \quad \text{snd} \quad v = \langle (\mathfrak{K}(m) - 1)m + 1, \mathfrak{K}(m)m \rangle.$$

We have

$$\mu^{2\ell-3}(01) = \mu^{2\ell-3}(\mathbf{t}_4\mathbf{t}_5) = \langle 3 \cdot 2^{2\ell-3} + 1, 5 \cdot 2^{2\ell-3} \rangle = wvz$$

for some words w and z. Observe that $|w| = (\mathfrak{K}(m) - 1)m - 3 \cdot 2^{2\ell-3} = rm + 2^{\ell-1}$. Since $\mu^{2\ell-3}(01) = \mu^{2\ell-3}(\mathbf{t}_1\mathbf{t}_2) = \langle 1, 2^{2\ell-3} \rangle$, we have $v = \langle rm + 2^{\ell-1} + 1, (r+1)m + 2^{\ell-1} \rangle$. If we set $a = rm + 1$ and $b = rm + 2^{\ell-1} + 1$, then $a < b \leq a + m$. It follows from the fact that $\mathbf{t}$ is overlap-free that $u \neq v$. This is a contradiction.

Assume now that $\mathfrak{K}(m') \leq \dfrac{2^{2\ell-2}}{m'}$. Let

$$u' = \langle r'm' + 1, (r'+1)m' \rangle \quad \text{and} \quad v' = \langle (\mathfrak{K}(m') - 1)m' + 1, \mathfrak{K}(m')m' \rangle.$$

Let $q = \lceil (r'm' + 1)/2^{\ell-2} \rceil$ and $H = \min\{(r'+1)m', (q+2)2^{\ell-2}\}$. Finally, put $U = \langle r'm' + 1, H \rangle$ and $V = \langle (r' + 2^{\ell-2})m' + 1, H + 2^{\ell-2}m' \rangle$. The word U is the prefix of u' of length $H - r'm'$. Because $\mathfrak{K}(m') = r' + 2^{\ell-2} + 1$, V is the prefix of v' of length $H - r'm'$. Since $u' = v'$, we must have $U = V$.

There are words w' and z' such that

$$\mu^{\ell-2}(\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}) = \langle (q-1)2^{\ell-2} + 1, (q+2)2^{\ell-2} \rangle = w'Uz'.$$

<table>
<tr><td colspan="3">$\langle (q-1)2^{\ell-2}+1, (q+2)2^{\ell-2} \rangle$</td><td></td><td colspan="3">$\langle (q+m'-1)2^{\ell-2}+1, (q+m'+2)2^{\ell-2} \rangle$</td></tr>
<tr><td>$\mu^{\ell-2}(\mathbf{t}_q)$</td><td>$\mu^{\ell-2}(\mathbf{t}_{q+1})$</td><td>$\mu^{\ell-2}(\mathbf{t}_{q+2})$</td><td></td><td>$\mu^{\ell-2}(\mathbf{t}_{q+m'})$</td><td>$\mu^{\ell-2}(\mathbf{t}_{q+m'+1})$</td><td>$\mu^{\ell-2}(\mathbf{t}_{q+m'+2})$</td></tr>
<tr><td>w'</td><td>U</td><td>z'</td><td></td><td>w''</td><td>V</td><td>z''</td></tr>
</table>

FIGURE 2. An illustration of the proof of Lemma 4.7.

Furthermore,

$$\mu^{\ell-2}(\mathbf{t}_{q+m'}\mathbf{t}_{q+m'+1}\mathbf{t}_{q+m'+2}) = \langle (q+m'-1)2^{\ell-2}+1, (q+m'+2)2^{\ell-2} \rangle = w''Vz''$$

for some words w'' and z''. Note that $0 \le |w'| = r'm' - (q-1)2^{\ell-2} = |w''| < 2^{\ell-2}$ (the inequalities follow from the definition of q). The suffix of $\mu^{\ell-2}(\mathbf{t}_q)$ of length $2^{\ell-2} - |w'|$ is a prefix of U. Similarly, the suffix of $\mu^{\ell-2}(\mathbf{t}_{q+m'})$ of length $2^{\ell-2} - |w''|$ is a prefix of V. Since $|w'| = |w''|$ and $U = V$, we must have $\mathbf{t}_q = \mathbf{t}_{q+m'}$. Similar arguments show that $\mathbf{t}_{q+1} = \mathbf{t}_{q+m'+1}$ and $\mathbf{t}_{q+2} = \mathbf{t}_{q+m'+2}$ (see Figure 2).

Now,

$$r' = \mathfrak{K}(m') - 2^{\ell-2} - 1 \le \frac{2^{2\ell-2}}{m'} - 2^{\ell-2} - 1 = \frac{2^{2\ell-3} - 5 \cdot 2^{\ell-2} - 3}{m'},$$

so $\dfrac{r'm'+1}{2^{\ell-2}} < 2^{\ell-1} - 5$. Therefore, $q + 4 < 2^{\ell-1}$. It follows that for each $j \in \{0,1,2\}$, the binary expansion of $q+m'+j-1$ has exactly one more 1 than the binary expansion of $q+j+2$. We find that $\mathbf{t}_{q+3}\mathbf{t}_{q+4}\mathbf{t}_{q+5} = \overline{\mathbf{t}_{q+m'}\mathbf{t}_{q+m'+1}\mathbf{t}_{q+m'+2}} = \overline{\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}}$. However, utilizing the fact that $\mathbf{t}$ is cube-free, it is easy to check that $X\overline{X}$ is not a factor of $\mathbf{t}$ whenever X is a word of length 3. This yields a contradiction when we set $X = \mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}$. □

Theorem 4.2. *Let $\gamma(k)$ be as in Definition 1.2. We have*

$$\liminf_{k\to\infty} \frac{\gamma(k)}{k} \le \frac{9}{10} \quad \text{and} \quad \limsup_{k\to\infty} \frac{\gamma(k)}{k} \le \frac{3}{2}.$$

Proof. For each positive integer ℓ, let $f(\ell) = \left\lfloor \dfrac{5 \cdot 2^{2\ell-3}}{3 \cdot 2^{\ell-2}+1} \right\rfloor$ and $h(\ell) = \left\lfloor \dfrac{2^{2\ell-2}}{2^{\ell-1}+3} \right\rfloor$. One may easily verify that $h(\ell) < f(\ell) \le h(\ell+1)$ for all $\ell \ge 3$. Lemma 4.7 informs us that $\mathfrak{K}(3\cdot 2^{\ell-2}+1) > f(\ell)$. This means that the prefix of $\mathbf{t}$ of length $(3 \cdot 2^{\ell-2}+1)f(\ell)$ is an $f(\ell)$-anti-power, so $\gamma(f(\ell)) \le 3 \cdot 2^{\ell-2}+1$. As a consequence,

$$\liminf_{k\to\infty} \frac{\gamma(k)}{k} \le \liminf_{\ell\to\infty} \frac{\gamma(f(\ell))}{f(\ell)} \le \lim_{\ell\to\infty} \frac{3 \cdot 2^{\ell-2}+1}{f(\ell)} = \frac{9}{10}.$$

Now, choose an arbitrary integer $k \ge 3$. If $h(\ell) < k \le f(\ell)$ for some integer $\ell \ge 3$, then the prefix of $\mathbf{t}$ of length $(3 \cdot 2^{\ell-2}+1)f(\ell)$ is an $f(\ell)$-anti-power. This implies that $\gamma(k) \le 3 \cdot 2^{\ell-2}+1$, so

$$\frac{\gamma(k)}{k} < \frac{3 \cdot 2^{\ell-2}+1}{h(\ell)}.$$

Alternatively, we could have $f(\ell) < k \le h(\ell+1)$ for some $\ell \ge 3$. In this case, Lemma 4.7 tells us that the prefix of $\mathbf{t}$ of length $(2^\ell+3)h(\ell+1)$ is an $h(\ell+1)$-anti-power. It follows that

$$\frac{\gamma(k)}{k} < \frac{2^\ell+3}{f(\ell)}$$

in this case.

Combining the above cases, we deduce that

$$\limsup_{k\to\infty}\frac{\gamma(k)}{k}\leq\limsup_{\ell\to\infty}\left[\max\left\{\frac{3\cdot 2^{\ell-2}+1}{h(\ell)},\frac{2^{\ell+1}+3}{f(\ell)}\right\}\right]=\max\left\{\frac{3}{2},\frac{6}{5}\right\}=\frac{3}{2}.$$

□

Remark 4.1. Preserve the notation from the proof of Theorem 4.2. We showed that

$$\frac{\gamma(k)}{k}<\frac{3\cdot 2^{\ell-2}+1}{h(\ell)}=\frac{3}{2}+o(1)$$

if $h(\ell)<k\leq f(\ell)$ and

$$\frac{\gamma(k)}{k}<\frac{2^{\ell}+3}{f(\ell)}=\frac{6}{5}+o(1)$$

whenever $f(\ell)<k\leq h(\ell+1)$ (the $o(1)$ terms refer to asymptotics as $k\to\infty$). This is indeed reflected in the top image of Figure 3, which portrays a plot of $\gamma(k)/k$ for $3\leq k\leq 2100$.

5. Concluding Remarks

In Theorems 3.1 and 3.2, we obtained the exact values of $\liminf\limits_{k\to\infty}(\Gamma(k)/k)$ and $\limsup\limits_{k\to\infty}(\Gamma(k)/k)$. Unfortunately, we were not able to determine the exact values of $\liminf\limits_{k\to\infty}(\gamma(k)/k)$ and $\limsup\limits_{k\to\infty}(\gamma(k)/k)$. Figure 3 suggests that the upper bounds we obtained are the correct values.

Conjecture 5.1. *We have*

$$\liminf_{k\to\infty}\frac{\gamma(k)}{k}=\frac{9}{10}\quad\text{and}\quad\limsup_{k\to\infty}\frac{\gamma(k)}{k}=\frac{3}{2}.$$

Recall that we obtained lower bounds for $\liminf\limits_{k\to\infty}(\gamma(k)/k)$ and $\limsup\limits_{k\to\infty}(\gamma(k)/k)$ by first showing that $\mathfrak{K}(m)\leq 2^{\lceil\log_2 m\rceil}(1+o(m))$. If Conjecture 5.1 is true, its proof will most likely require a stronger upper bound for $\mathfrak{K}(m)$.

We know from Theorem 3.1 that $(2\mathbb{Z}^+-1)\setminus\mathcal{F}(k)$ is finite whenever $k\geq 3$. A very natural problem that we have not attempted to investigate is that of determining the cardinality of this finite set. Similarly, one might wish to explore the sequence $(\Gamma(k)-\gamma(k))_{k\geq 3}$.

Recall that if w is an infinite word whose i^{th} letter is w_i, then $AP(w,k)$ is the set of all positive integers m such that $w_1w_2\cdots w_{km}$ is a k-anti-power. An obvious generalization would be to define $AP_j(w,k)$ to be the set of all positive integers m such that $w_{j+1}w_{j+2}\cdots w_{j+km}$ is a k-anti-power. Of course, we would be particularly interested in analyzing the sets $AP_j(\mathbf{t},k)$.

Define a (k,λ)-anti-power to be a word of the form $w_1w_2\cdots w_k$, where $w_1,w_2,\ldots,w_k$ are words of the same length and $|\{i\in\{1,2,\ldots,k\}\colon w_i=w_j\}|\leq\lambda$ for each fixed $j\in\{1,2,\ldots,k\}$. With this definition, a $(k,1)$-anti-power is simply a k-anti-power. Let $\mathfrak{K}_\lambda(m)$ be the smallest positive integer k such that the prefix of $\mathbf{t}$ of length km is not a (k,λ)-anti-power. What can we say about $\mathfrak{K}_\lambda(m)$ for various positive integers λ and m?

Finally, note that we may ask questions similar to the ones asked here for other infinite words. In particular, it would be interesting to know other nontrivial examples of infinite words x such that $\min AP(x,k)$ grows linearly in k.

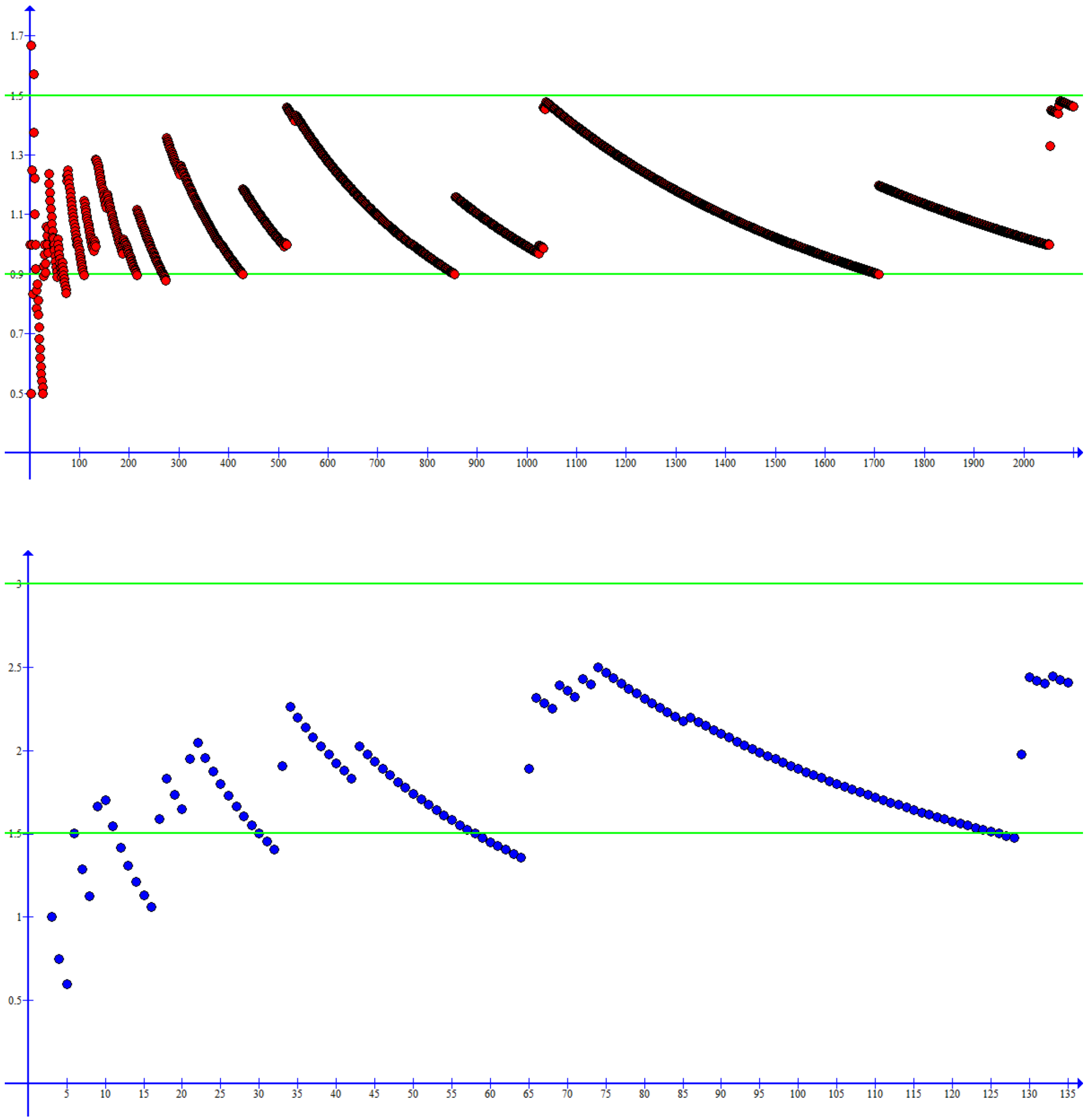

FIGURE 3. Plots of $\gamma(k)/k$ for $3 \le k \le 2100$ (top) and $\Gamma(k)/k$ for $3 \le k \le 135$ (bottom). In the top image, the green lines are at $y = 9/10$ and $y = 3/2$. In the bottom image, the green lines are at $y = 3/2$ and $y = 3$.

6. ACKNOWLEDGMENTS

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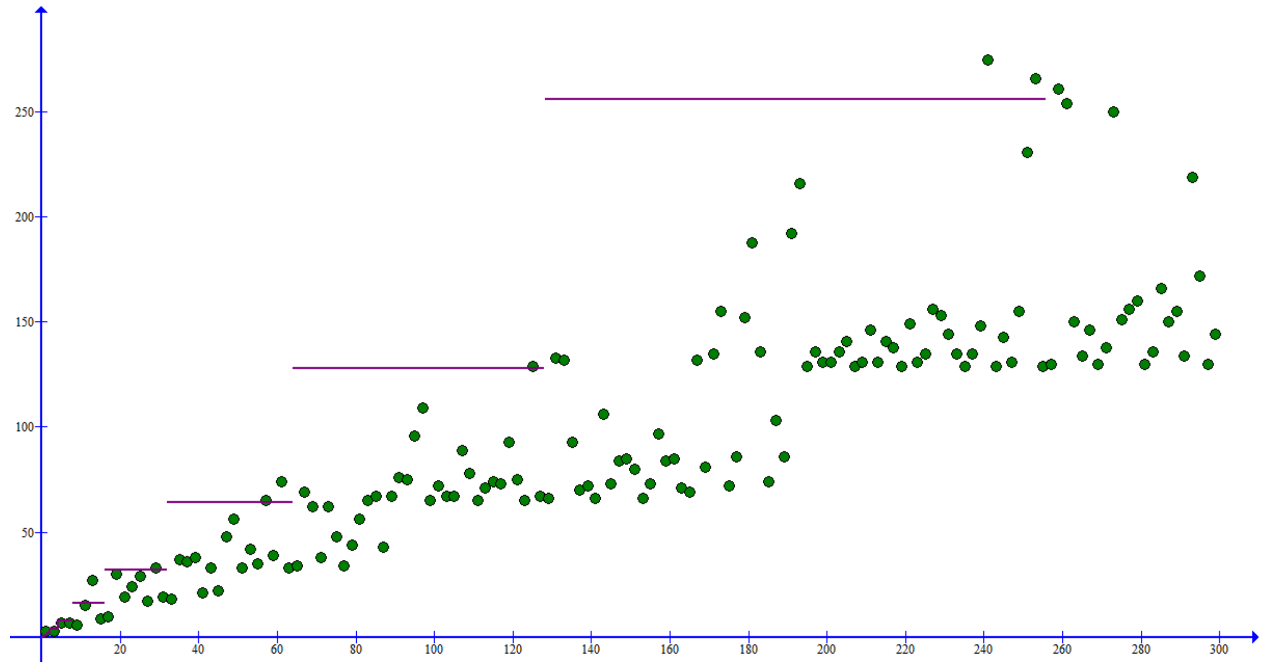

FIGURE 4. A plot of $\mathfrak{K}(m)$ for all odd positive integers $m \leq 299$. In purple is the graph of $y = 2^{\lceil \log_2 x \rceil}$.

REFERENCES

[1] J.-P. Allouche, Thue, Combinatorics on words, and conjectures inspired by the Thue-Morse sequence, *J. de Théorie des Nombres de Bordeaux*, 27, no. 2 (2015), 375–388.

[2] J.-P. Allouche and J. Shallit, The Ubiquitous Prouhet-Thue-Morse Sequence. *Sequences and their Applications: Proceedings of SETA '98*, (1999), 1–16.

[3] Y. Bugeaud and G. Han, A combinatorial proof of the non-vanishing of Hankel determinants of the Thue-Morse sequence, *Electronic Journal of Combinatorics* 21(3) (2014).

[4] S. Brlek, Enumeration of factors in the Thue-Morse word, *Discrete Applied Math.*, 24 (1989), 83–96.

[5] F. Dejean, Sur un theoreme de Thue. *J. Combinatorial Theory Ser. A* 13 (1972), 90–99.

[6] G. Fici, A. Restivo, M. Silva, and L. Zamboni, Anti-Powers in Infinite Words. *Proceedings of ICALP 2016*, (In Press).

[7] A. Thue, Über die gegenseitige Lage gleic her Teile gewisser Zeichenreihen, *Kra. Vidensk. Selsk. Skifter. I. Mat.-Nat. Kl.*, Christiana 1912, Nr. 10.